**Quantized Elastic Field**

Now let’s consider a simple classical example – an elastic field, which is basically the continuum limit of a set of harmonic oscillators [see CM folder for 1D example]. We’ll start by writing down the Hamiltonian.



where Y is the Young’s modulus of the medium, and μ is the (linear) mass density. With the usual canonical commutation relations:



There should be carrots on top of the φ’s and π’s, as they are operators, but I’ll leave this out for convenience…to get the excitations, it’s easiest to work out the time-development of an operator, say, φ(x). And this can be done via dφ/dt = i[H,φ] or by writing down the Lagrangian and doing the EL equations. Let’s do both:



(the two guys are zero because φ commutes with all other φ’s, and with differences too therefore) and another commutator,



So we have:



Or through the Lagrangian…



and we may verify that the formula for constructing H from L, analogous to the one used for multiple particles,



does indeed give us our H back (and note that time dependence doesn’t matter because H is constant for all time). For instance,



and since H is constant for all time, we can just say,



We may also, for future reference, write our commutation relations now between φ and , i.e.,



Note these are valid at any equal time, because time development preserves commutation relations. So I’ve left time out altogether – or rather, put t = 0, for convenience. So anyway, to get the equations of motion, we’d form the action



and take the functional derivative w/r to φ(x,t) and minimize it.



So our equation is again,



And for an elastic solid we’d also impose boundary conditions.



[assuming periodic boundary conditions, say]. To solve, we’d expand φ in terms of the eigenfunctions of the spatial part.



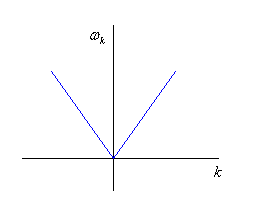
So we have to find its eigenfunctions.

They are clearly



where v is the velocity of waves in the solid. This is the conventional dispersion relation for harmonic waves in a solid, in the long wavelength limit [recall going to continuum screws up the real dispersion relation].



which is just the low wavelength limit of the discrete case. Plugging this into our equation of motion, it will reduce to



and the solution to this equation is done via usual ODE stuff. We get:



So that the general solution of the equation of motion is



We’re going to want some Q properties and commutation relations in just a second. So let’s work on those. First off, since φ is Hermitian, we must have:



and so we can identify,



And now we can work out the Q commutation relations by demanding they reproduce the φ commutation relations. I’ll do so at t = 0 for convenience,



Can see this necessitates,



Now we’re going to construct the ‘free-field expansion’ of φ. Recall from QM that we can identify operators which develop harmonically as annihilators/creators of energy excitations. So looking back at φ(x,t) we know now that the energy excitations are just ωk. And up to normalization, the annihilation/creation operators are:



We can determine the normalization constant by imposing:



So we have:



[noting ωk is symmetric w/r to k] And so now we can write:



So finally, where M is the total mass, we have:



Let’s plug these results into the Hamiltonian and see what we get. First note:



And so (can set t = 0 for simplicity here since H is constant)



and,



aaaand now we’ll use the commutation relations:



So we have:



just like we expect. We could try to work out the eigenstate wave-functionals for our field. I’ll just borrow our work in the QM folder/Many Particles/N 1D oscillators file.

A picture containing art

Description automatically generated with medium confidence

There we found, the wavefunction for N 1D oscillators obeying periodic boundary conditions over the length L = Na, was,



where,



and Rn = na is the position of the nth oscillator, assuming lattice spacing, *a*. Now we need to translate this to our situation, so there are some unfortunate cosmetic changes to make. I’m going to first change xn → φn, as φ is the name of our quantum mechanical variable, not x. And going to change Rn → xn, as now x is the variable which labels our positions. And we’ll note that our BZ = (-π/a, π/a) would now just go to (-∞, ∞), since a → 0. So at the moment, we’d have something like,



where,



Now consider the variable inside the Hermite polynomial. And note that this same variable appears within the Gaussian factor, squared. Let’s say Mion = μa, where μ is the mass density, and say L = Na, where L is the length of the chain of oscillators.



Now we can take the limit that a → 0, and convert the sum to an integral:



So then as long as our continuous elastic field loop has finite length L, we can say:



So this is our wave-functional. And it gives us the probability density that the field will attain some configuration φ(x). And this φ(x) is *not* an operator – it’s a number (a function). Just know that there are an infinite number of k = 2πn/L ‘s – one for every integer n. Also, that Re will just change all the e-ikx’s to cos(kx)’s, since φ(x) is presumed real. And that pre-factor is a little iffy, as it contains an a → 0. But that probably helps keep ψ finite, compensating for the product over the infinite # of k’s. Anyway, can see that the wavefunctional for the simplest quantum field – an elastic field – is pretty formidable. And one wouldn’t want to do anything that requires manipulating these things, in so far as possible. For instance, how would you normalize ψ? How would you compute expectations with it? How would you solve the Schrodinger differential-functional equation for it in the first place, if we didn’t have our discrete N-harmonic oscillator analogy to go on? Even still, we can use the wavefunctional to figure out the most probable field configuration for a given state. We’d just take the functional derivative w/r to φ(x) and set to 0. Looks like we should find φ(x) = 0 is the most probable configuration, at least in the ground state, corresponding to zero displacement of all the little field oscillators. But that would just be the most likely value. Even the ground state has non-zero energy, and so we see that the field will actually be making continual fluctuations about this value, just as a normal harmonic oscillator’s displacement oscillates about x = 0.

We can also get the time-development of the wavefunctional,

